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Stabilization of cycles for difference equations with a noisy PF control

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Abstract

For a difference equation $x_{n+1} = f(x_n)$ with a continuous f, increasing for small x and vanishing at zero, we apply a noise-involving proportional feedback control every kth step

$$x_{n+1} = \begin{cases} f\left((\nu + \ell_1 \chi_{m+1}) x_n\right) + \ell_2 \chi_{m+1}, & n = mk, \\ f(x_n), & n \neq mk, \end{cases} \quad x_0 > 0,\ m, n \in \mathbb{N}_0,\ k \in \mathbb{N},\ \nu \in (0, 1].$$

The purpose of getting a stable blurred k-cycle is achieved and illustrated with examples. Some generalizations are considered.

Key words: Stochastic difference equations; proportional feedback control; population dynamics models; stable cycles

1 Introduction

For an equation

$$x_{n+1} = f(x_n), \quad x_0 > 0, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \dots\} \tag{1}$$

with an unstable equilibrium, several control methods were developed in the literature, e.g. [6,7,8,10,11,17]. These methods include Proportional Feedback (PF) control in the deterministic [8] and stochastic [4] versions, Prediction-based control [1,10,11,17] and Target Oriented control [6,7]. Some of these methods were used to stabilize cycles rather than an equilibrium in [2,3,11]. Stochastic versions of these control methods, applied to stabilize a blurred equilibrium, were considered in [1,4]. In addition, there are control methods where stabilization is achieved by noise only, see the recent papers [5,9] and references therein. In the present paper, we concentrate on a stochastic version of PF control, applied to stabilize blurred cycles.

We consider the control by the proportional feedback (PF) method. This method, first introduced in [8], involves reduction of the state variable at each k-th step, proportional to the size of the variable

$$x_{n+1} = \begin{cases} f(\nu x_n), & n = mk, \\ f(x_n), & n \neq mk, \end{cases} \tag{2}$$

where $x_0 > 0$, $m, n \in \mathbb{N}_0$, $\nu \in (0, 1]$, $k \in \mathbb{N}$. However, the reduction coefficient may involve a stochastic component resulting in a multiplicative noise

$$x_{n+1} = \begin{cases} f\left((\nu + \ell_1 \chi_{m+1}) x_n\right), & n = mk, \\ f(x_n), & n \neq mk, \end{cases} \tag{3}$$

$x_0 > 0$, $m, n \in \mathbb{N}_0$, $\nu \in (0, 1]$, $k \in \mathbb{N}$. We can also con-

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sider the case when the reduction coefficient is deterministic but there are random fluctuations of x_n at the control step

$$x_{n+1} = \begin{cases} \max\left\{f(\nu x_n) + \ell_2 \chi_{m+1}, 0\right\}, & n = mk, \\ f(x_n), & n \neq mk, \end{cases} \qquad (4)$$

$x_0 > 0$, $n, m \in \mathbb{N}_0$, $\nu \in (0, 1]$, $k \in \mathbb{N}$. While (3) accounts for possible fluctuations of harvesting effort, model (4) considers a random deduction at each control step, which can describe either pollution and disturbance associated with a harvesting event, as well as consumption or sampling applied by controllers and independent of the current population size. On a positive side, harvesters may cause increase by unintentionally fostering immigration or boosting available food supply.

The deterministic version of cycle stabilization by PF control was justified in [2]. Stabilization of a positive equilibrium with PF method shifts an equilibrium closer to zero and is achieved in an interval $\nu \in (\alpha, \beta) \subset (0, 1)$: for smaller values of ν, zero becomes the only stable equilibrium, for higher values, a positive equilibrium still can be unstable. When we applied PF control on each kth step [2], it led to construction of an asymptotically stable k-cycle, with all the values between zero and a positive equilibrium. Here we construct a stochastic analogue of this process. Stabilization of stochastic equations with proportional feedback was recently explored in the continuous case [13], as well as the idea of periodic controls [18].

The paper is organized as follows. In Section 2, we introduce main assumptions for function f and stochastic perturbations, and discuss properties of a k-iteration of function f. Section 3 contains results on the existence of a blurred k-cycle in the presence of stochastic multiplicative perturbations of the control parameter ν when the level of noise ℓ is small, while Section 4 deals with the controlled equation for additive stochastic perturbations. Section 5 contains examples with computer simulations illustrating the results of the paper, along with some generalizations. In particular, a modification of PF method "centered" at an unstable equilibrium K instead of zero, is developed and applied to construct a blurred k-cycle in the neighborhood of K, when both stochastic, multiplicative and additive perturbations, are present.

2 Definitions and Auxiliary Results

In this paper, we impose an assumption on the map f in a right neighbourhood of zero.

Assumption 1 The function $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$, and there is a real number $b > 0$ such that $f(x)$ is strictly monotone increasing, while the function $f(x)/x$ is strictly monotone decreasing on $(0, b]$, $f(b) > b$, while $f(b)/b > f(x)/x$ for any $x \in (b, \infty)$.

Remark 1 *Note that, once Assumption 1 holds for a certain $b > 0$, it is also satisfied for any $b_1 \in (0, b]$.*

Many population dynamics maps [16] satisfy Assumption 1. We truncate values at zero when necessary, to satisfy $f : [0, \infty) \to [0, \infty)$, which is a common practice [14]. Examples include the Ricker model

$$x_{n+1} = f_1(x_n) = x_n e^{r(1-x_n)} \qquad (5)$$

for $r > 1$, with any $b \le 1/r$, the logistic model (truncated at zero)

$$x_{n+1} = f_2(x_n) = \max\left\{r x_n(1 - x_n), 0\right\}$$

for $r > 2$, with $b \le 1/2$. In these maps, f_i are unimodal, increasing on $[0, x_{\max}]$ and decreasing on $[x_{\max}, \infty)$, with the only critical point on $[0, \infty)$, which is a global maximum. However, Assumption 1 can hold for functions which have more than one critical point, for example, for the map developed in [12] to describe the growth of the bobwhite quail population

$$f_5(x) = x\left(A + \frac{B}{1 + x^\gamma}\right), \quad A, B > 0, \quad \gamma > 1, \qquad (6)$$

which, generally, has a maximum and a minimum, and is unbounded.

We denote by $(\Omega, \mathcal{F}, (\mathcal{F}_m)_{m\in\mathbb{N}}, \mathbb{P})$ a complete filtered probability space, $\chi := (\chi_m)_{m\in\mathbb{N}}$ is a sequence of independent random variables with the zero mean. The filtration $(\mathcal{F}_m)_{m\in\mathbb{N}}$ is naturally generated by the sequence $(\chi_m)_{m\in\mathbb{N}}$, i.e. $\mathcal{F}_m = \sigma\{\chi_1, \dots, \chi_m\}$. The standard abbreviation "a.s." is used for both "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. A detailed discussion of stochastic concepts and notation can be found in [15]. We consider (3) and (4), where the sequence $(\chi_m)_{m\in\mathbb{N}}$ satisfies the following condition.

Assumption 2 $(\chi_m)_{m\in\mathbb{N}}$ is a sequence of independent and identically distributed continuous random variables, with the density function $\phi(x)$ such that

$$\phi(x) > 0, \quad x \in [-1, 1], \quad \phi(x) \equiv 0, \quad x \notin [-1, 1].$$

Remark 2 *In fact, Assumption 2 can be relaxed to the condition $\mathbb{P}\{\chi \in [1 - \varepsilon, 1]\} > 0$ for any $\varepsilon > 0$, which would allow to include discrete distributions, where $\mathbb{P}\{\chi = 1\} > 0$.*

In numerical simulations, we also consider the combination of (3) and (4)

$$x_{n+1} = \begin{cases} f\left((\nu+\ell_1\chi_{m+1})x_n\right)+\ell_2\chi_{m+1}, & n=mk, \\ f(x_n), & n\neq mk, \end{cases}$$
$$x_0>0,\ m,n\in\mathbb{N}_0,\ k\in\mathbb{N},\ \nu\in(0,1]. \tag{7}$$

Let us start with some auxiliary results on $f^k(x) = f(f^{k-1}(x))$ and $g(x) := f^k(\nu x)$ for any $\nu \in (0,1]$, where Assumption 1 holds. Obviously $f : [0,b] \to [0,f(b)]$ is increasing and continuous, and there is an increasing and continuous inverse function $f^{-1} : [0,f(b)] \to [0,b]$. As $f(b) > b$ and $f(x)/x$ is decreasing on $(0,b]$ by Assumption 1, $f(x) > x$ for $x \in [0,b]$, and also f is increasing. Thus $f^{-1}(b) : [0,f(b)] \to [0,b]$ is well defined, and $f^{-1}(b) \in (0,b)$. Evidently $f^2 : [0,f^{-1}(b)] \to [0,f(b)]$ is continuous and increasing, since f is increasing on $[0,b]$, and $f^2(x) \in [0,f(b)]$ for $x \in [0,f^{-1}(b)]$. Therefore $f^{-2} : [0,f(b)] \to [0,f^{-1}(b)]$ is also well defined and increasing. Similarly, $f^{-k} : [0,f(b)] \to [0,f^{1-k}(b)]$ exists and is increasing for any $k \in \mathbb{N}$. Denote

$$b_j := f^{1-j}(b), \quad j\in\mathbb{N}, \quad j\neq 1, \quad b_1 = b, \tag{8}$$

then $f(b_{j+1}) = b_j$, $j = 1,2,\dots,k$, and

$$b = b_1 > b_2 > \cdots > b_k > 0. \tag{9}$$

Lemma 2.1 *If f satisfies Assumption 1, this assumption also holds for f^k with b_k instead of b, where b_k is defined in* (8).

Proof. The function $f : [0,b] \to [0,f(b)]$ is continuous and monotone increasing, so is $f^k : [0,b_k] = [0,f^{1-k}(b)] \to [0,f(b)]$. Next, let us prove that $f^k(x)/x$ is monotone decreasing on $[0,b_k]$. Let $0 < x_1 < x_2 \le b_k$. Then $f(x_1) \le b_{k-1}, \dots, f^j(x_1) \le b_{k-j}$, $j = 1,\dots,k-1$. Since $f(x)/x$ is decreasing on $[0,b]$, while f is increasing, $f(x_1)/x_1 > f(x_2)/x_2$, $f(f(x_1))/f(x_1) > f(f(x_2))/f(x_2)$, $\dots$, $f^k(x_1)/f^{k-1}(x_1) > f^k(x_2)/f^{k-1}(x_2)$ and

$$\frac{f^k(x_1)}{x_1} = \frac{f^k(x_1)}{f^{k-1}(x_1)} \cdots \frac{f^2(x_1)}{f(x_1)}\frac{f(x_1)}{x_1}$$

$$> \frac{f^k(x_2)}{f^{k-1}(x_2)} \cdots \frac{f^2(x_2)}{f(x_2)}\frac{f(x_2)}{x_2} = \frac{f^k(x_2)}{x_2}.$$

Also, $f(0) = 0$ implies $f^k(0) = 0$, and $f(b) > b$, by (9), yields that $f^k(b_k) = f^k\left(f^{1-k}(b)\right) = f(b) > b > b_k$.

Finally, let us justify that $f^k(x)/x < f^k(b_k)/b_k$ for any $x > b_k$ by induction. For $k = 1$, $f(x)/x < f(b)/b$ follows from Assumption 1.

For $k = 2$ and $x > f^{-1}(b) = b_2$, we consider two possible cases: $f(x) < b$ and $f(x) \ge b$. In the former case, $f(f(x)) < f(b)$, as f increases on $[0,b]$, and

$$\frac{f^2(x)}{x} < \frac{f(b)}{x} < \frac{f(b)}{b_2} = \frac{f^2(b_2)}{b_2}.$$

For $f(x) \ge b$, by Assumption 1, $f(x)/x < f(b_2)/b_2$ for any $x > b_2$, as $b_2 < b$, and $f(f(x))/f(x) \le f(b)/b$. Thus

$$\frac{f(f(x))}{x} = \frac{f(f(x))}{f(x)}\frac{f(x)}{x} < \frac{f(b)}{b}\frac{f(b_2)}{b_2}$$

$$= \frac{f(b)}{b}\frac{b}{b_2} = \frac{f(b)}{b_2} = \frac{f^2(b_2)}{b_2}.$$

Next, let us proceed to the induction step. Assume $\dfrac{f^n(x)}{x} < \dfrac{f^n(b_n)}{b_n} = \dfrac{f(b)}{b_n}$ for any $x > b_n$. Consider $x > b_{n+1}$. Then either $f^n(x) < b$ or $f^n(x) \ge b$. In the former case $f^n(x) < b$, we have $f(f^n(x)) < f(b)$ due to monotonicity of f on $[0,b]$ and

$$\frac{f^{n+1}(x)}{x} < \frac{f(b)}{x} < \frac{f(b)}{b_{n+1}} = \frac{f^{n+1}(b_{n+1})}{b_{n+1}}.$$

In the latter case $f^n(x) \ge b$ we get

$$\frac{f^{n+1}(x)}{x} = \frac{f(f^n(x))}{f^n(x)}\frac{f^n(x)}{x} < \frac{f(b)}{b}\frac{f^n(b_{n+1})}{b_{n+1}}$$

$$= \frac{f^{n+1}(b_{n+1})}{b}\frac{b}{b_{n+1}} = \frac{f^{n+1}(b_{n+1})}{b_{n+1}},$$

where in the inequality we used $\dfrac{f^n(x)}{x} < \dfrac{f^n(b_{n+1})}{b_{n+1}}$ for any $x > b_{n+1}$ by the induction assumption. Also, $f(u)/u \le f(b)/b$ for any $u = f^n(x) \ge b$ by Assumption 1, while equalities applied notation (8). This concludes the proof. $\square$

Define the function Ψ_k as

$$\Psi_k(z) := \frac{z}{f^k(z)}, \quad z\in(0,b_k), \quad k\in\mathbb{N}, \tag{10}$$

and formally introduce the limit

$$\Psi_k(0) := \lim_{x\to 0^+}\frac{x}{f^k(x)}. \tag{11}$$

Lemma 2.2 *Let Assumption 1 hold, $k \in \mathbb{N}$ and Ψ_k be defined as in* (10), (11). *Then*

(1) $\Psi_k : (0,b_k) \to \Big(\Psi_k(0), \Psi_k(b_k)\Big)$,

$\Psi_k^{-1} : \Big(\Psi_k(0), \Psi_k(b_k)\Big) \to (0,b_k)$;

(2) $0 \le \Psi_k(0) < \Psi_k(b_k) < 1$;

(3) *both Ψ_k and its inverse Ψ_k^{-1} are increasing and continuous on their domains.*

Proof. By Lemma 2.1, the function Ψ_k defined in (10) is increasing, continuous and hence has a unique inverse function on $(0, b_k)$. Following Assumption 1, we notice that the limit $\lim\limits_{x\to 0^+} \dfrac{f(x)}{x}$ exists (finite or infinite), is positive and greater than 1, since $f(x)/x$ is decreasing on $(0, b_k)$ and $f^k(b_k) > b_k$. Note that $\dfrac{f^k(x)}{x} = \dfrac{1}{\Psi_k(x)}$ and $\lim\limits_{x\to 0^+} \dfrac{f^k(x)}{x} = \lim\limits_{x\to 0^+} \dfrac{1}{\Psi_k(x)}$, where $\lim\limits_{x\to 0^+} \dfrac{1}{\Psi_k(x)} = 0$ if $\lim\limits_{x\to 0^+} \dfrac{f^k(x)}{x} = +\infty$. Thus (11) is well defined, and Part 1 is valid. As mentioned above, Ψ_k and its inverse are continuous monotone increasing in their domains and by Lemma 2.1, $\Psi_k(b_k) < 1$, which implies Parts 2 and 3. $\square$

To apply known results from [4], for each point $x^* \in (0, f(b))$, we are looking for the control parameter $\nu = \nu(x^*) \in (0, 1)$ such that x^* is a fixed point of the function $g(x) := f^k(\nu x)$. We recall from (8) that $b_1 = b$ and introduce

$$\begin{aligned} &\hat{x} = f^{-k}(x^*),\ \nu = \nu(x^*) := \Psi_k(f^{-k}(x^*)) \\ &\nu(x^*) = \Psi_k(\hat{x}),\ \hat{x} = \nu(x^*)x^* = \Psi_k^{-1}(\nu). \end{aligned} \tag{12}$$

Lemma 2.3 *Let Assumption 1 hold, $k \in \mathbb{N}$ and $x^* \in (0, f(b))$. The function $\nu(x^*)$ defined in (12) satisfies the following conditions:*

(1) *x^* is a fixed point of $g(x) := f^k(\nu x)$, i.e. $f^k\big(\nu(x^*)x^*\big) = x^*$, $\nu(x^*) = \Psi_k(\hat{x})$, $\hat{x} \in (0, b_k)$;*
(2) *$\nu(x^*) \in (\Psi_k(0),\ \Psi_k(b_k)) \subset (0, 1)$;*
(3) *$\nu(x^*)$ is an increasing function of x^* on $(0, f(b))$.*

Proof. 1. Let $x^* \in (0, f(b))$, then $f^{-k}(x^*) \in (0, b_k)$, thus $\Psi_k(f^{-k}(x^*))$ is well defined and $\nu(x^*) = \Psi_k(f^{-k}(x^*))$

$$= \frac{f^{-k}(x^*)}{f^k(f^{-k}(x^*))} = \frac{f^{-k}(x^*)}{x^*} = \frac{\hat{x}}{f^k(\hat{x})} = \Psi_k(\hat{x}),$$

hence $\hat{x} = \nu(x^*)x^* \in (0, b_k)$ and

$$f^k\big(\nu(x^*)x^*\big) = f^k\left(\frac{f^{-k}(x^*)}{x^*}x^*\right) = f^k\left(f^{-k}(x^*)\right) = x^*.$$

2. We have $x^* \in (0, f(b))$ and $\hat{x} = f^{-k}(x^*) \in (0, b_k)$. Thus Lemma 2.2, Part 1 implies $\nu(x^*) \in (\Psi_k(0),\ \Psi_k(b_k)) \subset (0, 1)$.
3. By Lemma 2.2 and Assumption 1, for any $k \in \mathbb{N}$, both Ψ_k and f^{-k} are increasing functions on $(0, b_k)$ and $(0, f(b))$, respectively. Therefore $\nu(x^*) = \Psi_k(f^{-k}(x^*))$ is increasing as a function of x^* on $(0, f(b))$, which concludes the proof. $\square$

3 Multiplicative perturbations

Consider the deterministic PF with variable intensity $\nu_m \in (0, 1]$, applied at each k-th step, for a fixed $k \in \mathbb{N}$,

$$x_{n+1} = \begin{cases} f(\nu_m x_n), & n = mk, \\ f(x_n), & n \neq mk, \end{cases} \quad x_0 > 0, \tag{13}$$

$m, n \in \mathbb{N}_0$. Investigation of (13) will allow to analyze corresponding stochastic equation (3) with a multiplicative noise. For each $x^* \in \big(0, f(b)\big)$, we establish the control $\nu = \nu(x^*)$ and define an interval such that a solution of (3) remains in this interval, once the level of noise ℓ is small enough. This method was applied, for instance, in [1].

Further, we apply the result obtained in [4] for

$$z_{m+1} = g\left(\nu_m z_m\right) = f^k\left(\nu_m z_m\right),\ z_0 > 0,\ m \in \mathbb{N}, \tag{14}$$

to explore stochastic equation (3) with a multiplicative noise.

For any μ_1, μ_2 such that

$$\Psi_k(0) < \mu_1 < \mu_2 < \Psi_k(b_k), \tag{15}$$

we define

$$y_1 := \Psi_k^{-1}(\mu_1), \quad y_2 := \Psi_k^{-1}(\mu_2). \tag{16}$$

Lemma 3.1 *[4, Lemma 3.1] Let Assumption 1 hold for f^k, $k \in \mathbb{N}$, μ_1 and μ_2 satisfy (15) and, for each $m \in \mathbb{N}$,*

$$\nu_m \in [\mu_1, \mu_2]. \tag{17}$$

Then, for any $z_0 > 0$ and ε, $0 < \varepsilon < \min\{y_1, b_k - y_2\}$, where y_1,y_2 are defined in (16), there is $m_0 = m_0(x_0, \varepsilon)$, $m_0 \in \mathbb{N}$, such that the solution z_n of equation (14) for any $m \geq m_0$ satisfies

$$\nu_m z_m \in \left(y_1 - \varepsilon, y_2 + \varepsilon\right). \tag{18}$$

Remark 3 *Lemma 3.1 actually states (see its proof in [4]) that, for a prescribed $k \in \mathbb{N}$, for a small $\varepsilon > 0$, once a solution of (13) satisfies $\nu_{km}x_{km} \in (y_1 - \varepsilon, y_2 + \varepsilon)$, $m \in \mathbb{N}$, all the subsequent k-iterates $\nu_{m+j}x_{(m+j)k}$, $j \in \mathbb{N}$, are also in this interval. This is also true for the results based on Lemma 3.1, in particular, for Lemma 3.3 and Theorem 3.4.*

Lemma 3.2 *Let Assumption 1 hold, μ_1, μ_2 satisfy (15) and, for each $m \in \mathbb{N}$, (17) be fulfilled. For any $x_0 > 0$ and $\varepsilon > 0$, there is $m_0 = m_0(x_0, \varepsilon) \in \mathbb{N}$ such that for $m \geq m_0$, the solution of (13) satisfies*

$$x_{mk+j} \in \left(f^j(y_1) - \varepsilon, f^j(y_2) + \varepsilon\right), \quad j = 1, \dots, k, \tag{19}$$

where y_1 *and* y_2 *are defined in* (16).

Proof. Note that $y_1, y_2 \in (0, b_k)$ and f^j are continuous and monotone increasing on $(0, b_k)$ for $j = 0, \dots, k-1$. Therefore for any $\varepsilon > 0$ there is an $\varepsilon_1 > 0$ such that

$$\begin{array}{l} u \in (y_1 - \varepsilon_1, y_2 + \varepsilon_1) \quad \Rightarrow \\ f^j(u) \in \left(f^j(y_1) - \varepsilon, f^j(y_2) + \varepsilon\right), \end{array} \quad j = 1, \dots, k. \tag{20}$$

Choose $z_0 = x_0$ and $\varepsilon_2 < \min\{y_1, b_k - y_2, \varepsilon_1\}$ instead of ε in Lemma 3.1. Then for $m > m_0$, by (18), $\nu_{mk} x_{mk} \in (y_1 - \varepsilon_2, y_2 + \varepsilon_2)$. Since, by (20), also
$x_{mk+1} = f(\nu_{mk} x_{mk}) \in (f(y_1) - \varepsilon, f(y_2) + \varepsilon), \dots,$
$x_{mk+k} = f^k(\nu_{mk} x_{mk}) \in \left(f^k(y_1) - \varepsilon, f^k(y_2) + \varepsilon\right),$
this implies (19) and concludes the proof. $\square$

Let us proceed to stochastic equation (3).

We start with an auxiliary result which follows from Lemma 3.2.

Lemma 3.3 *Let* $k \in \mathbb{N}$ *be fixed, Assumptions 1 and 2 hold,* Ψ_k *be defined in* (10), $x^* \in (0, f(b))$, $\nu = \nu(x^*)$ *be as in* (12), *and*

$$\ell \in \left(0, \min\left\{\Psi_k(b_k) - \nu,\ \nu - \Psi_k(0)\right\}\right), \tag{21}$$

$$\underline{y} := \Psi_k^{-1}(\nu - \ell),\ \overline{y} := \Psi_k^{-1}(\nu + \ell),\ 0 < \underline{y} < \overline{y} < b_k. \tag{22}$$

Let x_n *be a solution to equation* (3) *with* ν,ℓ *satisfying* (12) *and* (21), *respectively.*

Then, for any $\varepsilon > 0$ *there is a* $m_0 = m_0(\varepsilon, x^*, x_0) \in \mathbb{N}$ *such that, for all* $m \ge m_0$, $m \in \mathbb{N}$,

$$x_{mk+j} \in \left(f^j(\underline{y}) - \varepsilon, f^j(\overline{y}) + \varepsilon\right), \quad j = 1, \dots, k, \quad a.s.$$

Proof. Since $x^* < f(b_k) < f(b)$, Lemma 2.3 implies $\nu(x^*) = \Psi_k(f^{-k}(x^*)) \in (\Psi_k(0),\ \Psi_k(b_k))$. Thus the right segment bound $\nu - \Psi_k(0)$ in (21) is positive. By Assumption 2 we have, a.s.,

$$\nu_m = \nu + \ell\chi_{m+1} \le \nu + \ell, \quad \nu_m = \nu + \ell\chi_{m+1} \ge \nu - \ell$$

and $\nu_m = \nu + \ell\chi_{mk+1} \ge \nu - \ell$, thus $\nu_m \in [\nu - \ell, \nu + \ell]$, a.s. Let

$$\mu_1 := \nu - \ell, \quad \mu_2 := \nu + \ell.$$

With ν as in (12) and ℓ satisfying (21), we have

$$\Psi_k(0) - \ell < \mu_1 < \mu_2 < \Psi_k(b_k) + \ell,$$

then, by Lemma 3.2, the statement of the lemma is valid. $\square$

Lemma 3.3 implies the main result of this section, which states that for any $k \in \mathbb{N}$ and $x^* \in (0, f(b))$, we can find a control ν and a noise level ℓ, such that the solution eventually reaches some neighbourhood of a k-cycle, a.s., and stays there.

Theorem 3.4 *Let Assumptions 1 and 2 hold,* Ψ_k *be defined in* (10), (11), $x^* \in (0, f(b))$ *be an arbitrary point,* $\nu = \nu(x^*)$ *be denoted in* (12), $\underline{y}$ *and* $\overline{y}$ *be defined in* (22), $x_0 > 0$ *and* $\ell \in \mathbb{R}$ *satisfy inequality* (21). *Then for the solution* x_n *of equation* (3), *the following statements hold.*
(i) For each $\varepsilon > 0$ *there exists a nonrandom* $m_0 = m_0(\varepsilon, x^*, x_0) \in \mathbb{N}$ *such that, for all* $m \ge m_0$,

$$x_{mk+j} \in \left(f^j(\underline{y}) - \varepsilon, f^j(\overline{y}) + \varepsilon\right), \quad j = 1, \dots, k, \quad a.s.$$

(ii) $\liminf\limits_{n \to \infty} x_{mk+j} \ge f^j(\underline{y})$, $\limsup\limits_{n \to \infty} x_{mk+j} \le f^j(\overline{y})$, $j = 1, \dots, k$, *a.s.*

Proof. Note that from condition (21) we have $\nu > \ell$. By Lemma 3.3, for any $x_0 > 0$ and $\varepsilon > 0$, there is $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that, a.s., $x_{mk+j} > f^j(\underline{y}) - \varepsilon$, $x_{mk+j} < f^j(\overline{y}) + \varepsilon$, $n \ge N_0$, $j = 1, \dots, k$, which immediately implies (i).

Choosing a sequence of $\varepsilon_n \to 0$ as $n \to \infty$, $n \in \mathbb{N}$ in (ii) and getting $m_0(\varepsilon_n) \in \mathbb{N}$, we deduce (ii). $\square$

Next, let us assume that the level of noise can be chosen arbitrarily small. Theorem 3.5 below confirms the intuitive feeling that, as the noise level ℓ is getting smaller, the solution of stochastic equation (3) behaves similarly to the solution of corresponding deterministic equation (2) in terms of approaching its stable cycle $\{f^j(\hat{x})\}$, $j = 1, \dots, k$, where $\hat{x}$ is defined in (12).

Theorem 3.5 *Let Assumptions 1 and 2 hold,* $k \in \mathbb{N}$ *be fixed,* $\hat{x} \in (0, b_k)$ *be an arbitrary point,* $x^* = f^k(\hat{x})$, $\nu = \nu(x^*)$ *be defined as in* (12), *and* $x_0 > 0$. *Then, for any* $\varepsilon > 0$, *there exists the level of noise* $\ell(\varepsilon) > 0$ *such that for each* $\ell < \ell(\varepsilon)$, *there is a nonrandom* $m_1 = m_1(\varepsilon, \ell, \hat{x}, x_0)$ *such that for the solution* x_n *of equation* (3) *with* $m \ge m_1$ *we have* $x_{mk+j} \in (f^j(\hat{x}) - \varepsilon, f^j(\hat{x}) + \varepsilon)$, $j = 1, \dots, k$, *a.s.*

Proof. First of all, from monotonicity of f^k notice that the map $x^* = f^k(\hat{x})$ is one-to-one, and an arbitrary $x^* \in (0, f(b))$ corresponds to a certain $\hat{x} \in (0, b_k)$. Next, by continuity of all f^j, for any $\nu = \nu(x^*)$ defined as in (12), there is a $\delta > 0$ such that

$$|x - \hat{x}| < \delta \Rightarrow \left|f^j(x) - f^j(x^*)\right| < \frac{\varepsilon}{2},\ j = 1, \dots, k. \tag{23}$$

Also, from the choice of ν in (12) and continuity of Ψ_k, there is $\ell(\varepsilon) > 0$ such that for $\ell < \ell(\delta)$,

$$\left|\underline{y} - \hat{x}\right| < \delta, \quad |\overline{y} - \hat{x}| < \delta,$$

since $\underline{y}$ and $\overline{y}$ defined in (22) continuously depend on ℓ. Thus, by (23),

$$\left|f^j(\hat{x}) - f^j(\underline{y})\right| < \frac{\varepsilon}{2}, \left|f^j(\overline{y}) - f^j(\hat{x})\right| < \frac{\varepsilon}{2}, \qquad (24)$$

$j = 1, \ldots, k$. Next, let us apply Theorem 3.4, Part (i), with $\frac{\varepsilon}{2}$ instead of ε. Then for all $m \geq m_0$, $j = 1, \ldots, k$, a.s.,

$$x_{mk+j} \in \left(f^j(\underline{y}) - \frac{\varepsilon}{2}, f^j(\overline{y}) + \frac{\varepsilon}{2}\right). \qquad (25)$$

In view of (24) and (25), $x_{mk+j} > f^j(\hat{x}) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = f^j(\hat{x}) - \varepsilon$ and $x_{mk+j} < f^j(\hat{x}) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = f^j(\hat{x}) + \varepsilon$, therefore $x_{mk+j} \in (f^j(\hat{x}) - \varepsilon, f^j(\hat{x}) + \varepsilon)$, $j = 1, \ldots, k$, a.s. $\square$

4 Additive perturbations

In this section we investigate similar problems for stochastic equation with additive perturbations (4), where f satisfies Assumption 1. Our purpose remains the same: to achieve pseudo-stabilization of a blurred cycle $\{f^j(x^*)\}$, $j = 1, \ldots, k$. Here x^* is an arbitrary point $x^* \in \big(0, f(b)\big)$.

Denoting again $g(x) := f^k(\nu x)$, we can connect (4) to the equation with $x_{mk} = z_m$, $x_0 = z_0$,

$$z_{m+1} = \max\left\{g(z_m) + \ell\chi_{m+1}, 0\right\}, \; x_0 > 0, \; n \in \mathbb{N}. \qquad (26)$$

Let $x^* \in (0, f(b))$, $\nu = \Psi_k^{-1}(x^*)$, $\hat{x} = \nu x^* = f^{-k}(x^*) \in (0, b_k)$. Note that $b_k/\nu > x^*$ and, for a fixed ν, by Lemma 2.1, $g(x) = f^k(\nu x)$ satisfies Assumption 1 for $\nu x \in (0, b_k]$, so $g(b_k/\nu)/(b_k/\nu) < g(x^*)/x^*$. Here the equality $g(x^*) = x^*$ is due to Lemma 2.3, Part 1. Thus $\frac{b_k}{\nu} - g\left(\frac{b_k}{\nu}\right) > 0$. In addition, $g(x) > x$ for $x \in (0, x^*)$ and $g(x) > x, x \in (x^*, b_k/\nu)$. For $\ell = 0$, from monotonicity of g on $(0, b_k/\nu)$, x^* is an attractor of g on $(0, b_k/\nu)$. Moreover, $g(x) < x$ for any $x > x^*$ implies x^* is an attractor for any $z_0 > 0$. Our purpose is to choose $\ell > 0$ small enough, to have $z_{m+1} \in (0, b_k/\nu)$, once z_m is in this interval. However, attractivity of a positive equilibrium in a deterministic case, in the presence of the zero equilibrium, does not imply that zero is a repeller in the stochastic case, see [5,9] and references therein. Assumption 2 and its generalized version in Remark 2 allow to make a conclusion on attractivity of x^*, a.s.

We choose $\delta_0 > 0$ satisfying

$$\delta_0 < \min\left\{\frac{b_k}{\nu} - g\left(\frac{b_k}{\nu}\right), \max_{x \in [0, x^*]}\left[g(x) - x\right]\right\}. \qquad (27)$$

Define the numbers y_1, y_2, $\hat{x}_1$, $\hat{x}_2$ as

$$\begin{aligned} y_1 &:= \sup\left\{x \in [0, x^*] \,|\, g(x) - x \geq \delta_0\right\}, \\ \hat{x}_1 &:= \nu y_1 \in (0, b_k), \; y_1 \in (0, x^*), \\ y_2 &:= \inf\left\{x \in [x^*, b_k/\nu] \,|\, g(x) - x \leq -\delta_0\right\}, \\ \hat{x}_2 &:= \nu y_2 \in (0, b_k), \; y_2 \in (x^*, \frac{b_k}{\nu}). \end{aligned} \qquad (28)$$

According to the choice of δ_0, the sets in (28) are nonempty, so y_1,y_2, $\hat{x}_1$ and $\hat{x}_2$ are well defined. Denote

$$y_3 = \inf\left\{x \in [x^*, \infty) \,|\, g(x) - \delta_0 \leq y_1\right\}, \qquad (29)$$

where y_3 is assumed to be infinite if the set in the right-hand side of (29) is empty. As stated in [4, Lemma 4.1], the numbers y_1, y_2 and y_3 defined by (28) and (29), respectively, exist.

Lemma 4.1 *[4, Theorem 4.5] Let Assumptions 1 and 2 hold, $x^* \in (0, f(b))$ be an arbitrary point, $\nu = \nu(x^*)$ be chosen as in (12), $g(x) = f^k(\nu x)$ and δ_0 satisfy (27). Suppose that y_1, y_2, y_3 are denoted in (28) and (29), respectively, and z_m is a solution to equation (26) with an arbitrary $z_0 > 0$ and $\ell > 0$ satisfying $\ell \leq \delta_0$. Then*
(i) for each $\varepsilon_1 > 0$, there exists a random $\mathcal{M}(\omega) = \mathcal{M}(\omega, x_0, \ell, x^*, \varepsilon_1)$ such that for $m \geq \mathcal{M}(\omega)$ we have, a.s. on Ω,

$$y_1 \leq z_m \leq y_2 + \varepsilon_1; \qquad (30)$$

(ii) for each $\varepsilon_1 > 0$ and $\gamma \in (0, 1)$, there is a nonrandom number $M = M(\gamma, x_0, \ell, x^*, \varepsilon_1)$ such that

$$\mathbb{P}\{y_1 \leq z_m \leq y_2 + \varepsilon_1, \text{ for } m \geq M\} > \gamma; \qquad (31)$$

(iii) we have $\liminf_{m\to\infty} z_m \geq y_1$, $\limsup_{m\to\infty} z_m \leq y_2$, a.s.

Another result that will be used in future is also stated below. It illustrates that a solution will eventually be in any arbitrarily small neighborhood of x^* with an arbitrarily close to 1 probability.

Lemma 4.2 *[4, Theorem 4.6] Let Assumptions 1 and 2 hold, $z_0 > 0$ be an arbitrary initial value, $x^* \in (0, f(b))$ be an arbitrary point, $\nu = \nu(x^*)$ be chosen as in (12).*

Then, for each $\varepsilon > 0$ and $\gamma \in (0, 1)$, we can find δ_0 such that for the solution z_m to (26) with $\ell \leq \delta_0$, and for some nonrandom $M = M(\gamma, x_0, \ell, x^*, \varepsilon) \in \mathbb{N}$, we have $\mathbb{P}\{z_m \in (x^* - \varepsilon, x^* + \varepsilon) \; \forall m \geq M\} \geq \gamma$.

This leads to two main results on stable blurred k-cycles of (4).

Theorem 4.3 *Let Assumptions 1 and 2 hold, $\hat{x} \in (0, b_k)$ be an arbitrary point, $x^* = f^k(\hat{x})$, $\nu = \nu(x^*)$ be chosen as in (12), and δ_0 satisfy (27). Suppose that $\hat{x}_1$ and $\hat{x}_2$ are defined as in (28), and x_n is a solution to (4) with an arbitrary $x_0 > 0$ and $\ell > 0$ satisfying $\ell \leq \delta_0$.*

Then (i) For any $\varepsilon > 0$, there exists a random $\mathcal{M}(\omega) = \mathcal{M}(\omega, x_0, \ell, \hat{x}, \varepsilon)$ such that for $m \geq \mathcal{M}(\omega)$ we have, a.s. on Ω, $f^j(\hat{x}_1) \leq x_{km+j} \leq f^j(\hat{x}_2) + \varepsilon$, $j = 1, \dots k$.
(ii) For each $\varepsilon > 0$ and $\gamma \in (0, 1)$, there is a nonrandom number $M = M(\gamma, x_0, \ell, \hat{x}, \varepsilon)$ such that

$$\mathbb{P}\{f^j(\hat{x}_1) \leq x_{km+j} \leq f^j(\hat{x}_2) + \varepsilon, \, m \geq M\} > \gamma, \quad (32)$$

$j = 1, \dots k$.
(iii) For a solution x_n of (4) we have

$$\liminf_{n\to\infty} x_{km+j} \geq f^j(\hat{x}_1), \quad \limsup_{n\to\infty} x_{km+j} \leq f^j(\hat{x}_2),$$
$$a.s, j = 1, \dots k. \quad (33)$$

$j = 1, \dots k$.

Proof. Recall from (28) that $\nu y_1 = \hat{x}_1$, $\nu y_2 = \hat{x}_2$. From continuity and monotonicity of f, for any $\varepsilon > 0$, there is a $\varepsilon_1 > 0$ such that (30) implies

$$f^j(\hat{x}_1) \leq f^j(\nu z_m) \leq f^j(\hat{x}_2) + \varepsilon, \quad j = 1, \dots, k. \quad (34)$$

We have

$$x_{mk} = z_m, \quad x_{mk+j} = f^j(\nu z_m), \quad j = 1, \dots, k. \quad (35)$$

(i) Choosing this ε_1 as in (i) of Lemma 4.1, we find $\mathcal{M}(\omega) = \mathcal{M}(\omega, x_0, \ell, x^*, \varepsilon_1)$ such that (30), and thus (32) are satisfied.
(ii) Further, (ii) in Lemma 4.1 implies for $M = M(\gamma, x_0, \ell, x^*, \varepsilon_1)$ inequality (31). Thus by (34) and (35) we have $\mathbb{P}\{f^j(\hat{x}_1) \leq x_{km+j} \leq f^j(\hat{x}_2) + \varepsilon, \text{ for } m \geq M\}$ $\geq \mathbb{P}\{\hat{x}_1 \leq z_m \leq y_2 + \varepsilon, \text{ for } m \geq M\} > \gamma$.
(iii) As x_{mk+j} and z_m are connected with (35), application of Part (iii) in Lemma 4.1 immediately implies (33). $\square$

Theorem 4.4 *Let Assumptions 1 and 2 hold, $x_0 > 0$, $\hat{x} \in (0, b_k)$ be an arbitrary point, $x^* = f^k(\hat{x})$, $\nu = \nu(x^*)$ be chosen as in (12).*

Then, for each $\varepsilon > 0$ and $\gamma \in (0, 1)$, we can find δ_0 such that for the solution x_n to (4) with $\ell \leq \delta_0$, and for some nonrandom $M = M(\gamma, x_0, \ell, \hat{x}, \varepsilon) \in \mathbb{N}$, $j = 1, \dots, k$, we have

$$\mathbb{P}\left\{x_{km+j} \in \left(f^j(\hat{x}) - \varepsilon, f^j(\hat{x}) + \varepsilon\right) \forall m \geq M\right\} \geq \gamma, \quad (36)$$

Proof. Let us choose ε_1 such that (34) is satisfied, fix $\gamma \in (0, 1)$ and find $M = M(\gamma, x_0, \ell, \hat{x}, \varepsilon_1) \in \mathbb{N}$ as in Lemma 4.2. Then $\mathbb{P}\{\nu z_m \in (\hat{x} - \varepsilon_1, \hat{x} + \varepsilon_1) \text{ for all } m \geq M\} \geq \gamma$, which by (35) implies (36). $\square$

5 Examples

We consider (7) combining multiplicative and additive noise. Similarly to the previous theorems, the following more general result can be obtained. However, the proof is long and technical and does not include any new ideas. Therefore we do not present it, but only illustrate stated below Proposition 1 with computer simulations.

Proposition 1 *Let Assumptions 1 and 2 hold, $x_0 > 0$, $\hat{x} \in (0, b_k)$ be an arbitrary point, $x^* = f^k(\hat{x})$, $\nu = \nu(x^*)$ be chosen as in (12). Then, for each $\varepsilon > 0$ and $\gamma \in (0, 1)$, we can find δ_1 and δ_2 such that for the solution x_n to (7) with $\ell_1 \leq \delta_1$, $\ell_2 \leq \delta_2$ and for some nonrandom $M = M(\gamma, x_0, \ell_1, \ell_2, \hat{x}, \varepsilon) \in \mathbb{N}$, we have $\mathbb{P}\left\{x_{km+j} \in \left(f^j(\hat{x}) - \varepsilon, f^j(\hat{x}) + \varepsilon\right) \forall m \geq M\right\} \geq \gamma$, $j = 1, \dots, k$.*

Now we present examples of application of noisy PF control method to create a stable equilibrium or stable k-cycle in the neighborhood of nonzero point K. In all case noises χ are continuous uniformly distributed on $[-1, 1]$, and in all the simulations five runs with the same initial value are illustrated.

Example 1 *Let us apply PF control to the Ricker model (5). For $r = 2.8$, the non-controlled map is chaotic. First, we consider (3) with $\nu = 0.002$, $\ell = 0.0001$, noise applied every third step. We observe a blurred stable 3-cycle, see 1, left. Next, we simulate additive noise as in (4). We observe a blurred stable 3-cycle with similar amplitudes for larger ℓ, see Fig. reffigure2a, middle. Finally, we combine multiplicative and additive noise as in (7) for Ricker model with f as in (5), see Fig. 1, right.*

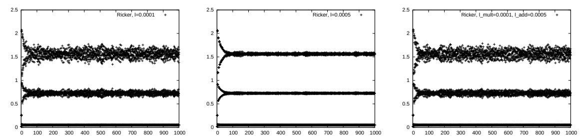

Fig. 1. Solutions of the Ricker difference equation with $f = f_1$ as in (5), $r = 2.8$, $k = 3$, $\nu = 0.002$ and (left) equation (3) with $\ell = 0.0001$, (middle) equation (4) with $\ell = 0.0005$, (right) equation (7) with $\ell_1 = 0.0001$, $\ell_2 = 0.0005$. Everywhere $x_0 = 0.5$.

Example 2 *Consider a particular case of (6), see [3,4],*

$$f(x) = x\left(0.55 + \frac{3.45}{1 + x^9}\right), \quad x \geq 0. \quad (37)$$

We apply PF with $k = 3$ to the three cases: the multiplicative noise, as in (3), the additive noise, as in (4), and the combined noise as in (7), see Fig. 2.

The standard PF control moves a positive equilibrium towards zero; applied at every kth step, it leads to a stable cycle in a right neighbourhood of zero. Now we modify this method choosing a positive equilibrium K_1 instead of zero. We apply PF control method to create a stable equilibrium or k-cycle in the nighboorhood of nonzero point K_1. The non-shifted PF control brings the state variable $1/\nu$ times closer to zero. We mimic

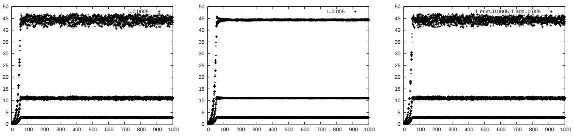

Fig. 2. Solutions of the difference equation with f as in (37), $k = 3$, $\nu = 0.02$ and (left) equation (3) with $\ell = 0.0005$, (middle) equation (4) with $\ell = 0.005$, (right) equation (7) with $\ell_1 = 0.0005$, $\ell_2 = 0.005$. Everywhere $x_0 = 0.5$.

this idea for a shifted version assuming that the state variable is proportionally moved to the fixed K_1. The controlled equation has the form $x_{n+1} = f(K_1 + \nu(x_n - K_1)) - K_1 + K_1 = f(\nu x_n + (1-\nu)K_1)$, $x_n \geq K_1$, $x_{n+1} = K_1 - [K_1 - f(K_1 - \nu(K_1 - x_n))] = f(\nu x_n + (1-\nu)K_1)$, $x_n \in (0, K_1)$,

$$x_{n+1} = f(\nu x_n + (1-\nu)K_1). \tag{38}$$

Example 3 *Define*

$$f(x) := \frac{9}{2}x^2(1-x), \quad x \in [0,1]. \tag{39}$$

The maximum value of f is achieved at $x_m = \frac{2}{3}$, $f(x_m) = \frac{2}{3}$, the inflection point is $x_0 = \frac{1}{3}$, $f''(x) > 0$ for $x \in (0, \frac{1}{3})$ and $f''(x) < 0$ for $x \in (\frac{1}{3}, 1)$, f has two positive equilibrium points $K_1 = \frac{1}{3}$, $K_2 = \frac{2}{3}$ and $f'\left(\frac{1}{3}\right) = \frac{3}{2} > 1$.

Consider a modification of PF method "centered" at $K_1 = 1/3$, see (38). It can be shown that, for $\nu \in (2/3, 1)$, equation (38) has two positive locally stable equilibrium points on both sides of K_1, each attracts a solution x_n with corresponding position of x_0 around K_1, see bifurcation diagram on Fig 3.

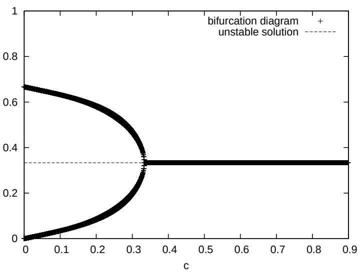

Fig. 3. Bifurcation diagram for (38) with f as in (39), $c = 1 - \nu$ changing from zero to 0.9 and x_0 changing from 0 to 1. We get an upper branch if x_0 changes from 1/3 to 1 and the lower branch if it changes from zero to 1/3.

Note that (38) is a particular case of Target Oriented Control [6], sufficient conditions for stabilization of K_1 in (38) were obtained in [7]. A modification of PF method is responsible for the left part of the diagram (bistability) while [7] gives an exact bound c^* such that for $c \in (c^*, 1)$, all solutions of (38) with $\nu := 1 - c$ and $x_0 \in (0,1)$ converge to $K_1 = 1/3$.

We introduce multiplicative noise in (38) to get for any $k \in \mathbb{N}$, $\nu \in (0,1]$,

$$x_{n+1} = \begin{cases} f((\nu + \ell_1\chi_{m+1})x_n + (1 - \nu - \ell_1\chi_{m+1})K_1), \\ n = mk, \ m, n \in \mathbb{N}_0, \ x_0 > 0, \\ f(x_n), \\ n \neq mk, \ m, n \in \mathbb{N}_0, \ x_0 > 0. \end{cases} \tag{40}$$

A multiplicative noise with small ℓ_1 does not change this type of behavior, as illustrated in Fig. 4. This also holds when coefficient ℓ_2 of the additive noise is relatively small and x_0 is relatively far from K_1, see Fig 5, left and middle. However, when ℓ_2 increases (in some limits), the solution started on the left of K_1 and close enough to K_1, is attracted to both equilibrium solutions, on the left and on the right of K_1, see Fig. 5, right. The same holds when $x_0 > K_1$.

Figure 6 illustrates construction of stable three-cycles when the initial value is taken on both sides of K_1.

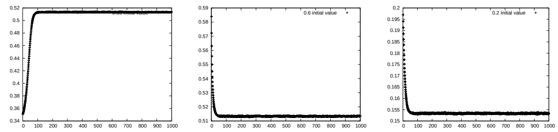

Fig. 4. Five runs of the difference equation with $f(x) = 4.5x^2(1-x)$, multiplicative noise with $\ell = 0.0005$, $\nu = 0.7$ and (left) $x_0 = 0.35$, (middle) $x_0 = 0.6$, (right) $x_0 = 0.2$.

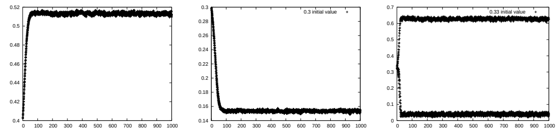

Fig. 5. For difference equation (40) with f as in (38) with additive noise (left) five runs for $\nu = 0.7$, $\ell = 0.001$, $x_0 = 0.4$, (middle) five runs for $\nu = 0.7$, $\ell = 0.001$, $x_0 = 0.3$, (right) six runs for $\nu = 0.8$, $\ell = 0.01$, $x_0 = 0.33$.

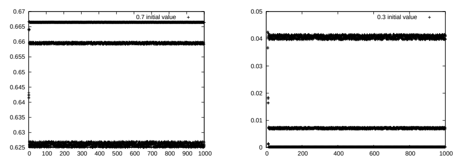

Fig. 6. Five runs of difference equation (40) with $f(x) = 4.5x^2(1-x)$, PF control applied every third step, multiplicative noise with $\ell_1 = 0.0001$, additive noise with $\ell_2 = 0.001$, $\nu = 0.7$ and (left) $x_0 = 0.3$, (right) $x_0 = 0.7$.

Example 4 *Define now*

$$f(x) := 6x^2(1-x), \quad x \in [0,1], \tag{41}$$

which has a positive equilibriums $K_1 \approx 0.211 < 1/3$. Note that function f, given by (41), satifies Sections 3-4

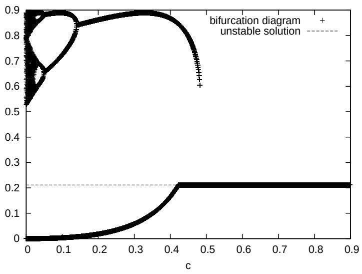

Fig. 7. Bifurcation diagram with $1-\nu$ changing from zero to 0.9 for the map $f(x) = 6x^2(1-x)$.

only on the left of K_1. It is illustrated by the bifurcation diagram, see Figure 7.

Fig. 8 illustrates a construction of a stable 2-cycle with multiplicative and additive noise. The left-side pictures, where the initial value $x_0 < K_1$, show a 2-cycle, while the right-side pictures, where $x_0 > K_1$, produce a 3-cycle.

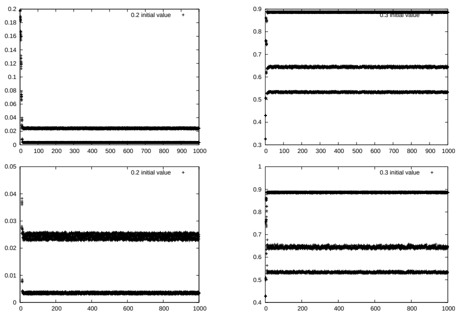

Fig. 8. Five runs of difference equation (40) with $f(x) = 6x^2(1-x)$, PF control applied every second step, multiplicative noise with $\ell_1 = 0.001$, additive noise with $\ell_2 = 0.01$, $\nu = 0.7$ and (left) $x_0 = 0.2$, (right) $x_0 = 0.3$. Lower row: five runs of the difference equation with $f(x) = 6x^2(1-x)$, PF control applied every second step, multiplicative noise with $\ell_1 = 0.01$, $\nu = 0.7$ and (left) $x_0 = 0.2$, (right) $x_0 = 0.3$.

6 Summary and discussion

First of all, numerical simulations show less restrictive conditions on ν in (38) than for classical (non-shifted) PF control. If we denote $c := 1-\nu$ in (38) then it becomes a particular case of Target Oriented Control with an unstable equilibrium K_1 as a target [6,7].

Possible generalizations and extensions of the present research include the following topics.

(a) Everywhere in simulations we assumed uniform continuous distribution, and all the estimates were dependent only on the noise amplitude. Specific estimates for particular types of noise distribution can be established.

(b) Everywhere we investigated asymptotic properties of solutions. However, analysis of so called transient behaviour, describing the speed of this convergence, starting from the initial point, maximal amplitudes for given initial values and noise characteristics, is interesting for applications.

(c) The present study can be extended to the case when unbounded, for example, normal distributions are involved.

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